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A COMPUTATION SCHEME FOR THE ASYMPTOTIC NUSSELT NUMBER IN DUCTS OF ARBITRARY CROSS-SECTION

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Abstract—A method is presented to approximate the asymptotic Nusselt number in long ducts with parallel walls and arbitrary cross-sections: The flow in the ducts is laminar and fully developed. The temperature of the ducts' walls changes in the form of a step. The Nusselt number is obtained for large distances from the location of the temperature step. The method shows how to obtain both upper and lower bounds to the Nusselt number and how to improve the approximation to any desired degree. Two examples are given: the circular duct (which is just the Graez problem, solved in [1]) and the square rectangular duct. An extension is made to cases where only numerical solutions are possible.

NOMENCLATURE

A,	area of duct's cross-section;
a_i,	coefficients in series solution;
$B(x, y)$,	boundary of duct's cross-sections;
b,	side of square duct;
C_p,	specific heat at constant pressure;
$D(x, y)$,	region inside cross-section;
f,	an approximation to x, y dependence of temperature, i.e. to θ;
f_0,	a solution to the first variational problem;
g,	a solution to the second variational problem;
h,	coefficient of heat convection;
k,	coefficient of thermal conduction;
Nu,	Nusselt number;
P,	pressure;
Pr,	Prandtl number;
Re,	Reynolds number;
r,	radius of circular duct;
r_0,	hydraulic radius;
S,	circumference of duct's cross-section;
T,	non-dimensional temperature;
t,	temperature;
t_0,	undisturbed temperature;
t_w,	wall temperature for $z \geqslant 0$;
u,	a function satisfying $u \geqslant w$;
W, $W(x, y)$,	velocity component in the z direction;
$\overline{W}$,	average velocity;
w,	non-dimensional velocity, $w = W/\overline{W}$;
x, y, z,	Cartesian coordinates, z parallel to the duct's generators;
α,	thermal diffusivity;
ζ,	non-dimensional z coordinate;
η,	non-dimensional y coordinate;
θ,	a function separated from the temperature as $T = \theta(\xi, \eta)\exp(-\lambda^2\zeta)$;
θ_i,	the ith iteration for θ in numerical solution;
K^2,	a lower bound to λ_0^2;
Λ^2,	an upper bound to λ_0^2;
Λ_0^2,	a solution to the first variational problem;
λ^2,	eigenvalue;
λ_0^2,	smallest eigenvalue;

μ,	viscosity;
μ_0,	a solution to the second variational problem;
ν,	kinematic viscosity;
ξ,	non-dimensional x coordinate;
ρ,	non-dimensional radius;
∇^2,	two-dimensional operator, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

INTRODUCTION

THIS paper considers a duct with parallel walls in which there is a fully developed laminar flow parallel to the ducts generators. The temperature of the fluid and of the ducts' walls is constant, say t_0. At $z = 0$ the temperature of the walls changes to t_w in the form of a step, and remains so for $z \geqslant 0$ (z being parallel to the ducts walls). Heat-transfer rates are sought for large z. This problem is generally known as the Graez problem and solutions for particular cross-sections are presented in [1–3]. This paper presents a method of solution for arbitrary cross-sections. This is done by the introduction of an auxiliary variational problem and then by the use of the Rayleigh–Ritz method to approximate the solution and to obtain upper bounds to the Nusselt number. A method to obtain lower bounds is also shown, but these do not come out to be as close to the exact values as the upper bounds do. Finally a numerical iteration scheme is suggested for cases where the boundaries of the ducts are such that no known functions are available for the Rayleigh–Ritz method.

ANALYSIS

Let a duct be defined by its boundaries $B(x, y)$. The z axis is taken parallel to the duct walls. The flow in the duct is fully developed and is given by $W(x, y)$ which satisfies the momentum equation:

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial Z} = \text{const.} \tag{1}$$

$$W = 0 \quad \text{on} \quad B(x, y).$$

The solution of equation (1) is assumed to be known (numerically even).

The temperature field is described by the energy equation:

$$W(x, y)\frac{\partial t}{\partial z} = \alpha\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) \tag{2}$$

$$t = t_0 \quad \text{at} \quad z \leqslant 0$$

$$t = t_w \quad \text{on} \quad B(x, y) \quad \text{at} \quad z > 0.$$

Equation (2) can be written now in a non-dimensional form by the use of the following definitions:

$$\iint_{B+D} \mathrm{d}x\,\mathrm{d}y = A \qquad \oint_B \mathrm{d}s = S \qquad r_0 = 2\frac{A}{S}$$

$$\xi = \frac{x}{r_0} \qquad \eta = \frac{y}{r_0} \qquad \overline{W} = \frac{1}{A}\iint_{B+D} W\,\mathrm{d}x\,\mathrm{d}y$$

$$w(\zeta, \eta) = \frac{W}{\overline{W}} \qquad Pr = \frac{\nu}{\alpha} \qquad Re = \frac{2r_0\overline{W}}{\nu}$$

$$\zeta = \frac{z}{r_0\,Pr\,Re} \qquad T = \frac{t - t_w}{t_0 - t_w}. \tag{3}$$

Equation (2) now becomes:

$$w(\xi, \eta)\frac{\partial T}{\partial \zeta} = 2\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2}\right) \tag{4}$$

$$T = 1 \quad \text{at} \quad \zeta \leqslant 0$$

$$T = 0 \quad \text{on} \quad B(\xi, \eta) \quad \text{at} \quad \zeta \geqslant 0.$$

The solution of equation (4) is known for special cross-sections, such as circular regions, annulus, two parallel plates; they are considered extensions of the Graez problem [1–3]. This paper proposes to obtain the asymptotic Nusselt number for arbitrary cross-sections.

Direct separation of variables yields for T the form:

$$T = \sum_{i=0}^{\infty} a_i \exp(-\lambda_i^2 \zeta)\,\theta_i(\xi, \eta) \tag{5}$$

and the coefficients a_i can be chosen to satisfy

$$\sum_{i=0}^{\infty} a_i\,\theta_i(\xi, \eta) = 1.$$

In general the eigenvalues λ_i^2 form an increasing series and for large values of ζ

$$\exp\left[-(\lambda_1^2 - \lambda_0^2)\zeta\right] \ll 1. \tag{6}$$

Let the Nusselt number be defined:

$$Nu = \frac{2r_0 h}{k}. \tag{7}$$

From which a short calculation yields:

$$Nu = \tfrac{1}{2}\lambda_0^2 \tag{8}$$

or, in another form

$$h = \frac{k}{4r_0}\lambda_0^2. \tag{9}$$

(Note that (8) holds even if $\lambda_1^2 = \lambda_0^2$, which is not the case here.) A variational problem can be set:

Find $f \in C^{(2)}(B + D)$, $f = 0$ on $B(\xi, \eta)$, such that the functional

$$\Lambda^2 = 2\frac{\iint\limits_D \nabla f \cdot \nabla f \,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint\limits_D wf^2 \,\mathrm{d}\xi\,\mathrm{d}\eta} \tag{10}$$

attains its minimum. The existence of this minimum is guaranteed (e.g. [4, 5]) for any $B(\xi, \eta)$ which can be divided into a finite number of smooth curves, i.e. for any physically possible cross-section. The resulting Euler–Lagrange equation for f is

$$-\Lambda^2 wf = 2\nabla^2 f. \tag{11}$$

Comparison of equation (11) and equation (4) shows that once the f which solves the variational problem is found, say f_0 with the corresponding Λ_0^2, then

$$f_0 = \theta, \qquad \Lambda_0^2 = \lambda_0^2. \tag{12}$$

It is proposed therefore to obtain approximations for both θ and λ_0 by the application of the Rayleigh–Ritz method to equation (10). Besides being an approximation, this would always yield an upper bound for h [see equation (9)], as of course

$$\Lambda^2 \geqslant \lambda_0^2. \tag{13}$$

Furthermore, assume the variational problem solved; then

$$\Lambda_0^2 = \min\left(\frac{2\iint\limits_D \nabla f \cdot \nabla f \,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint\limits_D wf^2 \,\mathrm{d}\xi\,\mathrm{d}\eta}\right) = 2\frac{\iint\limits_D \nabla f_0 \cdot \nabla f_0 \,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint\limits_D wf_0^2 \,\mathrm{d}\xi\,\mathrm{d}\eta}. \tag{14}$$

Let a function $u(\xi, \eta)$ be chosen such that

$$u(\xi, \eta) \geqslant w(\xi, \eta) \qquad \text{in} \quad (D + B). \tag{15}$$

Denote

$$K^2 = \frac{2\iint\limits_D \nabla f_0 \cdot \nabla f_0 \,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint\limits_D uf_0^2 \,\mathrm{d}\xi\,\mathrm{d}\eta}. \tag{16}$$

Equations (14–16) yield at once

$$K^2 \leqslant \Lambda_0^2. \tag{17}$$

Now set another variational problem:

Find $g \in C^{(2)}(D + B)$, $g = 0$ on $B(\xi, \eta)$, such that the functional

$$\mu^2 = 2\frac{\iint\limits_D \nabla g \cdot \nabla g \,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint\limits_D ug^2 \,\mathrm{d}\xi\,\mathrm{d}\eta} \tag{18}$$

attains its minimum. The Euler–Lagrange equation for this problem is

$$-\mu^2 ug = 2\nabla^2 g \tag{19}$$

and here u is chosen such that besides satisfying (15) it makes equation (19) simple to admit an exact solution. The solution of this variational problem is g_0 with the corresponding μ_0^2 and obviously

$$\mu_0^2 \leqslant K^2 \leqslant \Lambda_0^2. \tag{20}$$

A lower bound for $\lambda_0^2 = \Lambda_0^2$, and therefore for h, is found.

EXAMPLES

As an illustration of the method two examples are computed:

Example (*a*): *The circular duct*, [1]. r_0 is just the radius of the duct. Denote

$$W = 2\overline{W}(1 - \rho^2).$$

Equation (5) becomes

$$-\lambda^2(1 - \rho^2)\theta = \frac{d^2\theta}{d\rho^2} + \frac{1}{\rho}\frac{d\theta}{d\rho}.$$

A simple f, to be used in equation (10) is

$$f = 1 - \rho^2 + C(1 - \rho^3)$$

where the first power in ρ is not used because it gives a non-vanishing derivative at $\rho = 0$.

$$\frac{df}{d\rho} = -2\rho - 3C\rho^2$$

$$\iint_D (-2\rho - 3C\rho^2)^2 \rho \, d\rho \, d\theta = \frac{2\pi}{2520}(2520 + 6048C + 3780C^2)$$

$$\iint_D (1 - \rho^2)[1 - \rho^2 + C(1 - \rho^3)]^2 \rho \, d\rho \, d\theta = \frac{4\pi}{2520}(315 + 712C + 405C^2).$$

Substitution in equation (10) yields

$$\Lambda^2 = \frac{2520 + 6048C + 3780C^2}{315 + 712C + 405C^2}.$$

This expression attains its minimum for $C = -0{\cdot}513$. With this C, Λ^2 becomes:

$$\Lambda^2 = 7{\cdot}32$$

which may now be compared with $\lambda_0^2 = 7{\cdot}317$ taken from [1].

As an illustration, a lower bound is now found for λ_0^2. A simple form for u which satisfies equation (15) is

$$u = 2 \geqslant 2(1 - \rho^2) = w.$$

Equation (19) becomes

$$\rho^2 \frac{d^2 g}{d\rho^2} + \rho \frac{dg}{d\rho} + \mu^2 \rho^2 g = 0.$$

This is a Bessel equation of zero order with the solution

$g = J_0(\rho\mu)$ and as $g = 0$ at $\rho = 1$ μ_0 is just the first root of

$$J_0: \quad \mu_0 \approx 2{\cdot}4, \quad \mu_0^2 = 5{\cdot}76 < \lambda_0^2.$$

Of course the lower bound will not be in general

because it is found through the assumed $u(\xi, \eta)$ without any elaborate process like the Rayleigh–Ritz method.

Example (b): The square rectangular duct, [6]. Let the side of the square be b. Let the origin of the coordinates be at the lower left corner of the duct. The solution of equation (1) is

$$W = \frac{16b^2}{\mu\pi^4}\frac{dp}{dz}\sum_{m,n} \frac{1}{mn(m^2 + n^2)} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{b}$$

where both m and n are odd.

The coefficients of the first, second, third and fourth terms in the series are, respectively:

$$\frac{1}{2}, \quad \frac{1}{30}, \quad \frac{1}{30}, \quad \frac{1}{130}$$

and only the first three terms are retained:

$$W = \frac{16b^2}{\mu\pi^4}\frac{dp}{dz}\left[\frac{1}{2}\sin\frac{\pi x}{b}\sin\frac{\pi y}{b} + \frac{1}{30}\left(\sin\frac{\pi x}{b}\sin\frac{3\pi y}{b} + \sin\frac{3\pi x}{b}\sin\frac{\pi y}{b}\right)\right]$$

The definitions of equation (3) become:

$$A = b^2, \quad S = 4b, \quad r_0 = \frac{b}{2}, \quad \xi = 2\frac{x}{b},$$

$$\eta = 2\frac{y}{b}, \quad \overline{W} = -\frac{94}{45\pi^2} + \frac{16b^2}{\mu\pi^4}\frac{dp}{dz}$$

$$\frac{W}{\overline{W}} = w = \frac{45\pi^2}{94}\left[\frac{1}{2}\sin\frac{\pi\xi}{2}\sin\frac{\pi\eta}{2} + \frac{1}{30}\left(\sin\frac{\pi\xi}{2}\sin\frac{3\pi\eta}{2} + \sin\frac{3\pi\xi}{2}\sin\frac{\pi\eta}{2}\right)\right]$$

$$Re = -\frac{94}{45\pi^2} \times \frac{16b^3}{\nu\mu\pi}\frac{dp}{dz}$$

$$\zeta = \frac{45\pi^6\mu\alpha z}{752b^4(dp/dz)}.$$

A function f is now assumed:

$$f = \sin\frac{\pi\xi}{2}\sin\frac{\pi\eta}{2} + C\left(\sin\frac{\pi\xi}{2}\sin\frac{3\pi\eta}{2} + \sin\frac{3\pi\xi}{2}\sin\frac{\pi\eta}{2}\right).$$

(Note that the series solution of equation (1) always supply functions for the Rayleigh–Ritz method here.) These f and w are now substituted in equation (10) to yield:

$$\Lambda^2 = 10{\cdot}298\frac{1 + 10C^2}{1{\cdot}7304 - 1{\cdot}0376C + 2{\cdot}8542C^2}$$

This expression is differentiated with respect to C to yield the smallest value of Λ^2 at

$$C = -\frac{1}{28{\cdot}21} \qquad \text{which is} \quad \Lambda^2 = 5{\cdot}89.$$

Here also a lower bound for λ_0^2 may be of interest. A simple form for u which satisfies equation (15) is

$$u = \frac{45\pi^2}{94}\left(\frac{1}{2} - \frac{2}{30}\right) = \frac{39\pi^2}{188} \geqslant w.$$

Equation (19) becomes

$$-\frac{39\pi^2}{188}\mu^2 g = 2\nabla^2 g.$$

This is the Helmholtz equation with the solution

$$g = \sin\frac{\pi\xi}{2}\sin\frac{\pi\eta}{2}$$

(satisfying $g = 0$ at $\xi = 0$, $\eta = 0$, $\eta = 0$ and at $\xi = 2$).

Hence

$$\mu_0^2 = \frac{\pi^2}{39\pi^2/188} = \frac{188}{39} = 4{\cdot}821 < \lambda_0^2.$$

Of course, as in the case of the circular duct, the exact value is expected to be much nearer the upper bound.

COMPUTATION METHOD FOR DUCTS OF ARBITRARY CROSS-SECTIONS

An extension can now be made to ducts with such cross-sections that even the solution for W, equation (1), cannot be written in terms of known functions. In such cases equation (1) would be solved numerically. Assuming this done, the following iteration method can be used:

(a) Assume any $\theta_i(x, y) \not\equiv 0$ satisfying $\theta_i = 0$ on the boundaries.

(b) Compute

$$\frac{1}{\alpha}\lambda_i^2 = \frac{\iint (\nabla\theta_i)\,.\,(\nabla\theta_i)\,\mathrm{d}x\,\mathrm{d}y}{\iint W\theta_i^2\,\mathrm{d}x\,\mathrm{d}y}, \text{ numerically.}$$

(c) Solve

$$\nabla^2\theta_{i+1} = -W\frac{\lambda_i^2}{\alpha}\theta_i \qquad \text{for} \quad \theta_{i+1},$$

numerically, with $\theta_{i+1} = 0$ on the boundaries.

(d) Change $i + 1$ to i, i.e. set the values of θ_{i+1} instead of those of θ_i, then go back to step (b) above.

This procedure converges [7, 8]. If iteration is stopped before convergence, λ_i^2 is an upper bound for λ_0^2 [see variational formulation and equation (10)].

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Résumé—On présente une méthode d'approximation du nombre de Nusselt asymptotique dans de longs tuyaux à parois parallèles et à sections droites arbitraires: l'écoulement dans les tuyaux, est laminaire et entièrement établi. La variation, de la température des parois du tuyau a la forme d'un échelon. Le nombre de Nusselt est obtenu pour de grandes distances de l'endroit du saut de température. La méthode montre comment obtenir à la fois les bornes supérieure et inférieure du nombre de Nusselt et comment améliorer l'approximation à n'importe quel niveau désirable. Deux exemples sont donnés: le tuyau circulaire (qui est justement le problème de Graez, résolu dans [1]), et le tuyau à section carrée. On a étendu la méthode aux cas où seules les solutions numériques sont possibles.

Zusammenfassung—Zur Approximation der asymptotischen Nusselt-Zahl in langen Kanälen mit parallelen Wänden und beliebigen Querschnitten wird eine Methode angegeben. Die Kanalströmung sei laminar und voll ausgebildet. Die Temperature der Kanalwände ändert sich schrittweise. Die Nusselt-Zahl wird erhalten für grosse Abstände vom Ort des Temperatursprungs. Die Methode zeigt, wie sich sowohl obere als auch untere Grenzen der Nusselt-Zahl erhalten lassen und wie die Näherung auf beliebige Genauigkeit verbessert werden kann. Zwei Beispiele werden angegeben: Der Kanal mit Kreisquerschnitt (das ist das Graetz-Problem und ist in [1] gelöst) und mit Rechteckquerschnitt. Für Fälle, in welchen nur numerische Lösungen möglich sind, wurde eine Erweiterung gemacht.

Аннотация—Предложен метод определения асимптотических значений числа Нуссельта в длинных трубах с параллельными стенками и произвольным поперечным сечением. Поток в трубах предполагается ламинарным и полностью развитым. Температура стенок изменяется ступенчато. Число Нуссельта получено для больших расстояний от точки, где происходит температурный скачок. Метод позволяет получить нижние и верхние границы значений числа Нуссельта и улучшить приближение с любой необходимой степенью точности. Приводятся два примера: круглая труба (задача Гретца, решенная в [1]) и квадратный канал. Сделано обобщение на случаи, в которых возможны только численные решения.